and consistent gluing

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We extend the lattice gauge theory-type derivation of the Barrett-Crane spin foam model for quantum gravity to other choices of boundary conditions, resulting in different boundary terms, and re-analyze the gluing of 4-simplices in this context. This provides a consistency check of the previous derivation. Moreover we study and discuss some possible alternatives and variations that can be made to it and the resulting models.

I. INTRODUCTION

Spin foam models [1–3], and the Barrett-Crane one [4,5] in particular, are promising candidates for the construction of a quantum theory of gravity from a covariant perspective, implementing in a purely algebraic fashion the path integral or sum-over-geometries approach. Different versions of the Barrett-Crane model are present in the literature [6,8–10], all sharing the same amplitude for the vertices of the spin foam but using different ampitudes for its edges, leading to models with different physical properties (in particular, the perturbative finiteness of one of these versions [8,19,20] is due to the particular form of the edge amplitude).

In [11] a derivation of the Barrett-Crane model was given, showing how it can be obtained from a discretized BF theory, imposing at the quantum level (as projectors in the partition function) the analogue of the constraints that reduce BF theory to gravity in the Plebanski formulation of GR. The version obtained is the Perez-Rovelli version, thus shown to come naturally from a discretization of (constrained) BF with usual methods from lattice gauge theory, being originally derived from a field theory over a group [8]. In order to obtain the exact form of the edge amplitudes, i.e. the amplitude for the tetrahedra dual to the edges of the spin foam, the procedure used was to derive first the expression for the partition function corresponding to a single 4-simplex, taking into account all the necessary boundary terms, and then to glue 4-simplices along tetrahedra in their boundary, ending up with the partition function assigned to the whole triangulated manifold built up from them. The amplitude to be assigned to the edges (tetrahedra) results from this gluing procedure only and does not require any additional input or choice. The fact that it comes directly from the gluing is to be expected since it should encode the information describing the (geometric) interaction between 4-simplices. The advantage of this lattice gauge theory type of derivation compared with other existing derivations (being of course strongly related, see in particular the “connection formulation” of field theories over a group manifold [7]), is, in our opinion, that it makes the link between the Barrett-Crane spin foam model and the classical Plebanski action [15–18] more clear, and makes the analogies between gravity and lattice gauge theory more explicit. Moreover, it helps us to understand better the origin and the geometric meaning of the edge amplitudes in the partition function, and may also help to clarify the differencies between the various existing versions of the Barrett-Crane model. On the other hand, this approach has the shortcoming of being limited to a fixed triangulation of spacetime, while the field theory over a group allows to sum over all the triangulations, even if much remains to be understood about this sum.

In this letter, we extend the derivation of [11] to other choices of boundary conditions, following an analogous study for 3-dimensional gravity [14], obtain the corresponding boundary terms in the partition function for a single 4-simplex and then apply again the gluing procedure to get the full partition function for the triangulated manifold. Apart from giving the correct boundary terms in this case, this serves as a consistency check for the previous derivation. In fact it is of course to be expected that the amplitudes for the elements in the interior of the manifold, the edge amplitudes in particular, should not be affected by the choice of boundary conditions in the 4-simplices (having boundaries) whose gluing produces them. The result is that the derivation in [11] is indeed consistent, and we get again the Perez-Rovelli version of the Barrett-Crane model. We then examine a few alternatives to the procedure used in [11], exploiting the freedom left by that derivation. In particular, we study the effect of imposing the projection over the simple representations also in the boundary terms, since this may (naively) recall the imposition of the simplicity constraints in the kinetic term in the field theory over the group manifold, leading to the DePietri-Freidel-Krasnov-Rovelli version of the Barrett-Crane model [6]. Instead, this leads in the present case to several drawbacks, as we discuss, and to a model which is not the DePietri-Freidel-Krasnov-Rovelli version and it is not consistent, in the sense specified above, with respect to different choices of boundary conditions. Moreover, we study and discuss the model that can be obtained by not imposing the gauge invariance of the edge amplitude (as required in [11]), since it was mentioned in [13], explaining why we do not consider it a viable version of the Barrett-Crane model, and finally the class of models that can be obtained by imposing the two projections (simplicity and gauge invariance) more that once. All the calculations in this paper will be performed explicitely for the Euclidean case, but are valid (or can be easily extended to) in the Lorentzian case as well, as we will discuss in the following.

II. DERIVATION OF THE BARRETT-CRANE SPIN FOAM MODEL: CONSTRAINING AND GLUING

Let us now recall the basic elements of the derivation in [11]. The starting point is the expression for the partition function for $SO(4)$ BF theory discretized on the 2-complex dual to a 4-dimensional triangulated manifold:

$$Z_{BF}(SO(4)) = \int_{SO(4)} dg \prod_{\sigma} \sum_{J_\sigma} \Delta_{J_\sigma} \chi^{J_\sigma}(\prod_{e} g_e) \tag{1}$$

where σ are the parts of the dual plaquettes associated to each 4-simplex, also called "wedges" in the literature [12] (see Fig.1), the sum is over the representations J of $SO(4)$ (given by two half-integer parameters (j,k)) here attached to each wedge (in such a way that wedges belonging to the same plaquette get the same representation attached to them), $\Delta_{J_\sigma} = (2j+1)(2k+1)$ is the dimension of the representation, $\chi^J(g)$ is the character of the group element g in the representation J, and the group variables are associated to the links of the 2-complex, so that for each wedge there is a group element assigned to it, given by the product of the group elements associated to the edges of its boundary (see Fig.1). Here the role of the B field is played by the representations J on the plaquettes of the 2-complex, and that of the connection by the group variables on its edges.

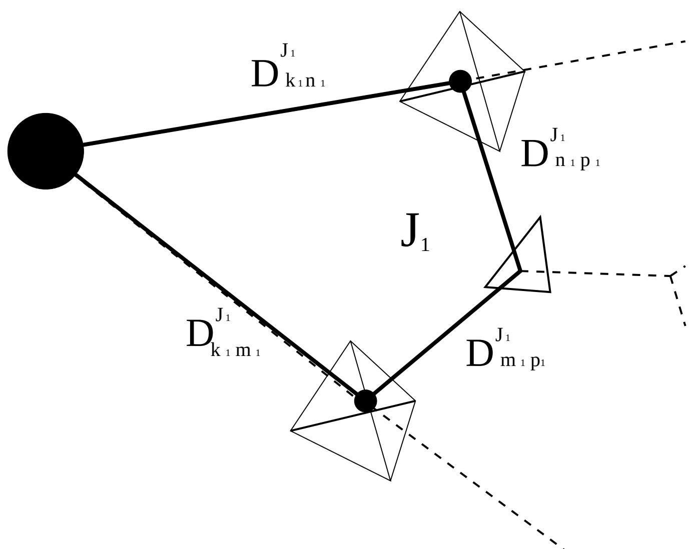

FIG. 1. Fig.1 - A wedge (the part of a dual face belonging to a single 4-simplex) with the D-functions for the group elements assigned to its boundary edges.

An analogous expression can be written in the Lorentzian case, with $SL(2,C)$ in place of $SO(4)$ and with the representations labeled by a continuous parameter, ρ, and a half-integer parameter, n (see [5,9]), and with "dimension" $(n^2 + \rho^2)$. Note that this partition function is only formally defined, since it is divergent, but it will give rise to a well-defined and convergent expression after the imposition of the Barrett-Crane constraints.

Now we consider the case of a single 4-simplex, consisting of 5 tetrahedra (which constitute its boundary), write the character explicitly in terms of the representation functions of the group elements assigned to each edge, choosing real representations, and sort the terms in the partition function per edge, to obtain:

$$Z_{BF}(SO) = \sum_{J_\sigma, \{k_e\}} \left(\prod_\sigma dim_{J_\sigma} \right) \prod_e A_e \left(\prod_{\tilde{e}} D \right)$$

$$= \sum_{\{J_\sigma\},\{k_e\}} \left(\prod_\sigma \Delta_{J_\sigma} \right) \prod_e \int_{Spin(4)} dg_e \, D^{J_1^e}_{k_{e1}m_{e1}} D^{J_2^e}_{k_{e2}m_{e2}} D^{J_3^e}_{k_{e3}m_{e3}} D^{J_4^e}_{k_{e4}m_{e4}} \left(\prod_{\tilde{e}} D^J_{il} \right) \quad . \tag{2}$$

There is a group element per edge, so that four representation functions coming from the four wedges (dual to triangles) incident to it are to be integrated. Each of these functions has two matrix indices, one referring to the vertex of the 2-complex (only one vertex since we are considering only one 4-simplex), and the other, referring to a tetrahedron on the boundary, contracted with one index of a D-function for an element attached to (and only to) a link which is exposed on the boundary (see Fig.1). The other index of each matrix for an exposed link (referring to a triangle) is contracted with the index coming from the D-function referring to the same triangle (again, see Fig.1).

It is crucial to note that the group elements attached to the links exposed on the boundary for each wedge are *not* integrated over, since we are working with fixed connection on the boundary, a boundary condition which can be easily shown to not require any additional boundary term in the classical action (see [14] for the 3-dimensional case).

We then pass from pure BF theory to 4-dimensional gravity imposing the Plebanski constraints on the B field [15–18] directly at the quantum level, i.e. as Barrett-Crane constraints on the representations J (or (n, ρ) in the Lorentzian case) labelling the wedges. In turn, this can be done imposing some projections on the edge amplitude in the partition function we have just described:

$$\begin{aligned} & A_e(GR) \, = \, P_g \, P_h \, A_e(BF) \, = \\ & = \int_{SO(4)} dg_1 D^{J_1}_{k_1 l_1}(g_1) D^{J_2}_{k_2 l_2}(g_1) D^{J_3}_{k_3 l_3}(g_1) D^{J_4}_{k_4 l_4}(g_1) \\ & \times \int_{SO(3)} dh_1 D^{J_1}_{l_1 i_1}(h_1) \int_{SO(3)} dh_2 D^{J_2}_{l_2 i_2}(h_2) \int_{SO(3)} dh_3 D^{J_3}_{l_3 i_3}(h_3) \int_{SO(3)} dh_4 D^{J_4}_{l_4 i_4}(h_4) \\ & \times \int_{SO(4)} dg_1' \, D^{J_1}_{i_1 m_1}(g_1') D^{J_2}_{i_2 m_2}(g_1') D^{J_3}_{i_3 m_3}(g_1') D^{J_4}_{i_4 m_4}(g_1'). \end{aligned} \tag{3}$$

The integrals in the $SO(3)$ h variables impose the simplicity of all the representations J, given by representations of the form (j, j), while additional integrals over $SO(4)$ restore the gauge invariance of the edge amplitudes, that, automatic in pure BF theory, is lost after the imposition of the simplicity constraints. These correspond to the simplicity and closure constraints in [4]. In the Lorentzian case everything works the same way, with $SL(2, C)$ and $SU(2)$ instead of $SO(4)$ and $SO(3)$ [5,9], with the simple representations given in this case by those labelled only by the continuous parameter ρ.

Note that we are *not* imposing any projection in the boundary terms, so that these are the same as those in pure BF theory. However, the projection over simple representations in the edge amplitude imposes automatically the simplicity also of the representations entering in the D-functions for the exposed edges.

Performing the integrals we get:

$$A_e = \frac{1}{\Delta_{J_1} \Delta_{J_2} \Delta_{J_3} \Delta_{J_4}} B^{J_1 J_2 J_3 J_4}_{k_1 k_2 k_3 k_4} B^{J_1 J_2 J_3 J_4}_{m_1 m_2 m_3 m_4}, \tag{4}$$

where the $B^{J_1 J_2 J_3 J_4}_{k_1 k_2 k_3 k_4} = \sum_J C^{J_1 J_2 J_3 J_4 J}_{k_1 k_2 k_3 k_4} = \sum_J \sqrt{\Delta_J} C^{J_1 J_2 J}_{k_1 k_2 k} C^{J_3 J_4 J}_{k_3 k_4 k}$ are the Barrett-Crane intertwiners, with the Cs being ordinary $SO(4)$ invariant tensors normalized such that the theta net is equal to one. The sets of indices of the intertwiners refer to the vertex of the 2-complex (and are contracted with others coming from the other edges) and to the boundary tetrahedra (and are contracted with the D-functions for the exposed edges) (see Fig.2). The factors in the denominator come from the simplicity projections.

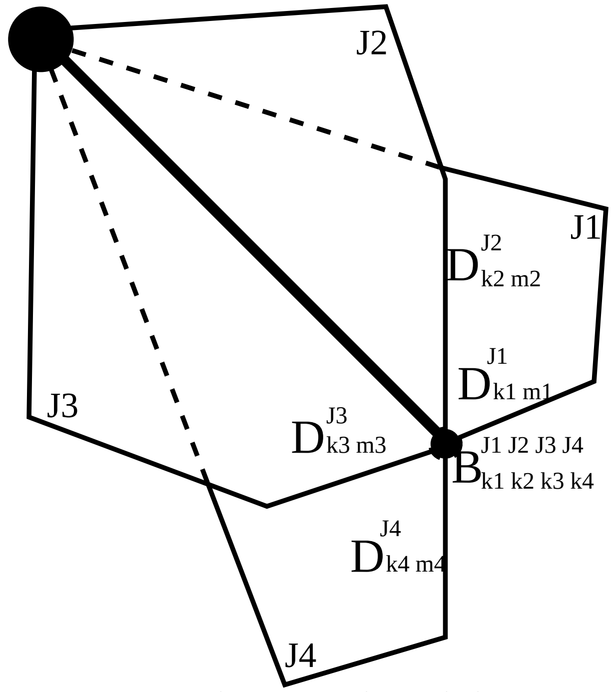

FIG. 2. Structure of the boundary term corresponding to a single tetrahedron, i.e. a single dual edge with the 4 wedges incident to it, and the corresponding 4 exposed edges.

Consequently the partition function for a single 4-simplex is:

$$Z_{BC} = \sum_{\{J_f\},\{k_{e'}\}} \prod_f \Delta_{J_f} \prod_{e'} \frac{B^{J_{e'1}J_{e'2}J_{e'3}J_{e'4}}_{k_{e'1}k_{e'2}k_{e'3}k_{e'4}}}{\Delta_{J_{e'1}}\Delta_{J_{e'2}}\Delta_{J_{e'3}}\Delta_{J_{e'4}}} \prod_v \mathbf{B}_{BC} \left(\prod_{\tilde{e}} D(\tilde{g})\right) \tag{5}$$

where the $\mathbf{B}_{BC}$ is the Barrett-Crane amplitude for the 4-simplex, and the boundary terms are given by one Barrett-Crane intertwiner for each tetrahedron on the boundary, and one D-function for each group element on each of the exposed edges, contacted with the intertwiner to form a group invariant (see Fig.2), plus a "regularizing" factor in the denominator.

The gluing of 4-simplices is now simply done by multiplying the partition functions for the individual 4-simplices, and integrating over the group variables that are not anymore on the boundary of the manifold, and required to be equal in the two 4-simplices, again because we are working with fixed connection on the boundary, so that the boundary data of the two 4-simplices being glued have to agree.

These group variables appear only in two exposed edges each, and the orthogonality between D-functions forces the representations corresponding to the two wedges to be equal:

$$\int_{SO(4)} dg D^J_{kl}(g)\, D^{J'}_{mn}(g) \;=\; \frac{1}{\Delta_J}\delta_{km}\delta_{ln}\delta_{JJ'}; \tag{6}$$

moreover, the factors $\frac{1}{\Delta_J}$ compensate for having two wedges corresponding to the same triangle, so that to each plaquette of the dual complex, or triangle of the triangulation, corresponds still only a factor Δ_J in the partition function. Finally, the equality of the matrix indices in the previous relation forces the Barrett-Crane intertwiners corresponding to the same shared tetrahedron to be fully contracted, so that the resulting amplitude for it (taking properly into account the normalization chosen above) is:

$$A_e \;=\; \frac{B^{J_1J_2J_3J_4}_{k_1k_2k_3k_4}B^{J_1J_2J_3J_4}_{k_1k_2k_3k_4}}{\left(\Delta_{J_1}\Delta_{J_2}\Delta_{J_3}\Delta_{J_4}\right)^2} = \frac{\Delta_{1234}}{\left(\Delta_{J_1}\Delta_{J_2}\Delta_{J_3}\Delta_{J_4}\right)^2} \tag{7}$$

where Δ_{1234} is the dimension of the space of intertwiners between the representations $J_1,..,J_4$, i.e. the number of possible intertwiners between these representations. Note that this is also the number of possible quantum tetrahedra for given values of their triangle areas, so it is really the most natural statistical weight for them in the partition function.

Note that the gluing is not trivial, in the sense that the end result is not just the product of pre-existing factors, but includes something resulting from the gluing itself (the factor Δ_{1234}).

In the end, the partition function we find for a general manifold with boundary, with fixed connection on the boundary, is:

$$Z_{BC} = \sum_{\{j_f\},\{k_{e'}\}} \prod_f \Delta_{j_f} \prod_{e'} \frac{B^{j_{e'1}j_{e'2}j_{e'3}j_{e'4}}_{k_{e'1}k_{e'2}k_{e'3}k_{e'4}}}{\Delta_{j_{e'1}}\Delta_{j_{e'2}}\Delta_{j_{e'3}}\Delta_{j_{e'4}}} \prod_e \frac{\Delta_{1234}}{\left(\Delta_{j_{e1}}\Delta_{j_{e2}}\Delta_{j_{e3}}\Delta_{j_{e4}}\right)^2} \prod_v \mathbf{B}_{BC} \left(\prod_{\tilde{e}} D(g_{\tilde{e}})\right) \tag{8}$$

where the $\{e'\}$ and the $\{e\}$ are the sets of boundary (incident to it) and interior edges of the spin foam, respectively, while the $\tilde{e}$ are the remaining exposed edges, where the boundary connection data are located. The partition function is then a function of the connection, i.e. of the group elements on the exposed edges. This is the Perez-Rovelli version of the Barrett-Crane model, with the appropriate boundary terms.

One can proceed analogously in the Lorentzian case, using the integral representation of the Barrett-Crane intertwiners (the resulting expression is of course more complicated, but with the same structure), and their formula for the evaluation of relativistic (simple) spin networks. All the passages above, in fact, amount to the evaluation of spin networks, which were proven to evaluate to a finite number, so the procedure above can be carried through similarly and sensibly.

We see that, starting from a ill-defined BF partition function, the imposition of the constraints has made the resulting partition function for gravity finite both in the Euclidean and Lorentzian cases [8,19,20].

III. FIXING THE BOUNDARY METRIC

Let us now study the case in which we choose to fix the B field on the boundary (i.e. by the metric field), and let us analyse first the classical action.

We note here that the partition function we will obtain in this section, being a function of the representations J (or ρ) of the group $SO(4)$ (or $SL(2,C)$ assigned to the boundary, and representing the B (metric) field, can be thought of as the Fourier transform [21] of the one we ended up with in the previous section, being instead function of the group elements, representing the connection field.

The $so(4)$ Plebanski action is:

$$S = \int_{\mathcal{M}} B \wedge F + \frac{1}{2}\phi B \wedge B \tag{9}$$

so that its variation si simply given by:

$$\delta S = \int_{\mathcal{M}} \delta B \wedge (F + \phi B) + \delta A \wedge (dB + A \wedge B + B \wedge A) - \int_{\partial\mathcal{M}} B \wedge \delta A, \tag{10}$$

and we see that fixing the connection on the boundary does not require any additional boundary term to give a well-defined variation, i.e. the field equations resulting from it are not affected by the presence of a boundary.

On the other hand, if we choose to fix the B field on the boundary, we need to introduce a boundary term in the action:

$$S = \int_{\mathcal{M}} B \wedge F + \frac{1}{2}\phi B \wedge B + \int_{\partial\mathcal{M}} B \wedge A \tag{11}$$

so that the variation leads to:

$$\delta S = \int_{\mathcal{M}} \delta B \wedge (F + \phi B) + \delta A \wedge (dB + A \wedge B + B \wedge A) + \int_{\partial\mathcal{M}} \delta B \wedge A, \tag{12}$$

and to the usual equations of motion.

Now we want to find what changes in the partition function for a single 4-simplex if we decide to fix the B field on the boundary, and then to study how the gluing proceeds in this case.

The additional term in the partition function resulting from the additional term in the action is $\exp \int_{\partial\mathcal{M}} B \wedge A$. We have to discretize it, expressing it in terms of representations J and group elements g on the boundary, and then multiply it into the existing amplitude. The connection terms on the boundary are then to be integrated out, since they are not held fixed anymore, while the sums over the representations have to be performed only on the bulk ones, i.e. only on the representations labelling the triangles in the interior of the manifold (none in the case of a single 4-simplex). A natural discretization [14] for the additional term is:

$$\exp \int_{\partial \mathcal{M}} B \wedge A = \prod_{\tilde{e}} \chi^{J}(g_{\tilde{e}}) = \prod_{\tilde{e}} D^{J}_{kk}(g_{\tilde{e}}) \tag{13}$$

where the representation J is the one assigned to a wedge with edges exposed on the boundary, and $g_{\tilde{e}}$ is actually the product $g_1 g_2$ of the group elements assigned to the two edges exposed on the boundary, and the product runs over the exposed parts of the wedges.

We multiply the partition function (5) by this extra term, and integrate over the group elements, simultaneously dropping the sum over the representations, since all the wedges are on the boundary, and thus all the representations are fixed.

Using again the orthogonality of the D-functions, eq. 6, the result is the following:

$$Z_{BC} = \prod_{f} \Delta_{J_f} \prod_{e'} \frac{B^{J_{e'1}J_{e'2}J_{e'3}J_{e'4}}_{k_{e'1}k_{e'2}k_{e'3}k_{e'4}}}{\left(\Delta_{J_{e'1}}\Delta_{J_{e'2}}\Delta_{J_{e'3}}\Delta_{J_{e'4}}\right)^{\frac{3}{2}}} \prod_{v} \mathbf{B}_{BC} \tag{14}$$

where also the k indices are fixed by the only constraint (coming again from the integration over the group above) that the ks appearing in different Barrett-Crane intertwiners but referring to the same triangle must be equal. Of course we see that the partition function is now a function of the representations J on the boundary and of their projections. The different power in the denominator of the boundary terms is necessary to have consistency in the gluing procedure, as we will see. Also, note that we did not impose any projection over the simple representations in the boundary terms, i.e. in the D-functions coming from the additional boundary term in the action, since we decided not to impose it in the D-functions for the exposed edges in (5) above. We will analyse the alternatives to this choice in the next section.

Now we proceed with the gluing of 4-simplices. The different 4-simplices being glued have to share the same boundary data for the common tetrahedron, i.e. the representations J and the projections k in the Barrett-Crane intertwiner referring to it have to agree. The gluing is performed again multiplying the partition functions for the two 4-simplices and summing over the ks, because they are now attached to a tetrahedron in the interior of the manifold. In this way the Barrett-Crane intertwiners corresponding to the same tetrahedron in the two 4-simplices get contracted with each other, and they give again a factor Δ_{1234} as before. The factors in the denominators of the (ex-)boundary terms are multiplied to give a factor $1/(\Delta_{J_1}\Delta_{J_2}\Delta_{J_3}\Delta_{J_4})^3$, but since we have a factor of Δ_{J_i} for each wedge and for each 4-simplex, the factor in the denominator of the amplitude for the interior tetrahedron is again $1/(\Delta_{J_1}\Delta_{J_2}\Delta_{J_3}\Delta_{J_4})^2$.

In the end the partition function for a generic manifold with boundary, with the boundary condition being that the metric field is fixed on it, is:

$$Z_{BC} = \sum_{\{J_f\}} \prod_{f} \Delta_{J_f} \prod_{e'} \frac{B^{J_{e'1}J_{e'2}J_{e'3}J_{e'4}}_{k_{e'1}k_{e'2}k_{e'3}k_{e'4}}}{\Delta_{J_{e'1}}\Delta_{J_{e'2}}\Delta_{J_{e'3}}\Delta_{J_{e'4}}} \prod_{e} \frac{\Delta_{1234}}{\left(\Delta_{J_{e1}}\Delta_{J_{e2}}\Delta_{J_{e3}}\Delta_{J_{e4}}\right)^2} \prod_{v} \mathbf{B}_{BC}. \tag{15}$$

It is understood that the sum over the representations Js is only over those labelling wedges (i.e. faces) in the interior of the manifold, the others being fixed.

We recall that this can be understood as the probabiblity amplitude for the boundary data, the representations of $SO(4)$ (or $SL(2,C)$) in this case or the $SO(4)$ group elements as in the previous section, in the Hartle-Hawking vacuum. If the boundary data are instead divided into two different sets, then the partition function represents the transition amplitude from the data in one set to those in the other. The Lorentzian case, again, goes similarly, with the same result.

We see that, apart from the boundary terms, we ended up again with the Perez-Rovelli version of the Barrett-Crane model. This was to be expected, since the bulk partition function should not be affected by the boundary conditions we have chosen for the single 4-simplices before performing the gluing, but the fact that this is indeed the case represents a good consistency check for the whole procedure we used to obtain the Barrett-Crane model from a discretized BF theory.

IV. EXPLORING ALTERNATIVES

Let us now go on to explore the alternatives to the procedure we have just used, to see whether there are other consistent procedures giving different results, i.e. different versions of the Barrett-Crane model. In particular we would like to see, for example, whether there is any variation of the procedure used above resulting in the DePietri-Freidel-Krasnov-Rovelli version of the Barrett-Crane model [6], i.e. the other version that can be derived from a field theory over a group manifold. Again, the analogous calculations in the Lorentzian case go through similarly.

We have seen in section II that two choices were involved in the derivation we performed: the way we imposed the constraints, with one projection imposing simplicity of the representations and the other imposing the invariance under the group of the edge amplitude, and the way we treated the D-functions for the exposed edges, i.e. without imposing any constraints on them. We will now consider alternatives to these choices, starting from the last one. A few other alternatives to the first were considered in [11].

A. Projections on the exposed edges

We then first leave the edge amplitude (3) as it is, and look for a way to insert an integral over the $SO(3)$ subgroup in the boundary representation functions. The idea of imposing the simplicity projections in the D-functions for the exposed edges may (naively) resemble the imposition of them in the kinetic term in the action for the field theory over a group, leading to the DePietri-Freidel-Krasnov-Rovelli version of the Barrett-Crane model [6], since in both cases there are precisely 4 of them for each tetrahedron, and they represent the boundary data to be transmitted across the 4-simplices (in the connection representation). Anyway, this is not the case, as we are going to show.

There are two possible ways of imposing the projections, corresponding to the two possibilities of multipliying the arguments of the D-functions by an $SO(3)$ element from the left or from the right, corresponding to projecting over an $SO(3)$ invariant vector the indices of the D-functions referring to the tetrahedra or those referring to the triangles (see figure3), then integrating over the subgroup as in (3), having for each boundary term:

$$
\begin{aligned}
&B^{J_1J_2J_3J_4}_{k_1k_2k_3k_4}D^{J_1}_{k_1m_1}(g_1)D^{J_2}_{k_2m_2}(g_2)D^{J_3}_{k_3m_3}(g_3)D^{J_4}_{k_4m_4}(g_4)\rightarrow\\
&\rightarrow B^{J_1J_2J_3J_4}_{k_1k_2k_3k_4}D^{J_1}_{k_1l_1}(g_1)D^{J_2}_{k_2l_2}(g_2)D^{J_3}_{k_3l_3}(g_3)D^{J_4}_{k_4l_4}(g_4)w^{J_1}_{l_1}w^{J_2}_{l_2}w^{J_3}_{l_3}w^{J_4}_{l_4}w^{J_1}_{m_1}w^{J_2}_{m_2}w^{J_3}_{m_3}w^{J_4}_{m_4}
\end{aligned}
\tag{16}
$$

or:

$$
\begin{aligned}
&B^{J_1J_2J_3J_4}_{k_1k_2k_3k_4}D^{J_1}_{k_1m_1}(g_1)D^{J_2}_{k_2m_2}(g_2)D^{J_3}_{k_3m_3}(g_3)D^{J_4}_{k_4m_4}(g_4)\rightarrow\\
&\rightarrow B^{J_1J_2J_3J_4}_{k_1k_2k_3k_4}w^{J_1}_{k_1}w^{J_2}_{k_2}w^{J_3}_{k_3}w^{J_4}_{k_4}w^{J_1}_{l_1}w^{J_2}_{l_2}w^{J_3}_{l_3}w^{J_4}_{l_4}D^{J_1}_{l_1m_1}(g_1)D^{J_2}_{l_2m_2}(g_2)D^{J_3}_{l_3m_3}(g_3)D^{J_4}_{l_4m_4}(g_4)
\end{aligned}
\tag{17}
$$

where in the first case the second set of invariant vectors is contracted with one coming from another boundary term, giving in the end no contribution to the amplitude, while in the second case there is a contraction between the Barrett-Crane intertwiners and these vectors, giving a different power in the denominator in (5), and the disappearence of the intertwiners from the amplitude.

Let us discuss the first case. The effect of the projection is to break the gauge invariance of the amplitude for a 4-simplex, and to decouple the different tetrahedra on the boundary. In fact the amplitude for a 4-simplex is then:

$$
Z=\sum_{\{J_f\},\{k_{e'}\}}\prod_f\Delta_{J_f}\prod_e\frac{B^{J_{e1}J_{e2}J_{e3}J_{e4}}_{k_{e1}k_{e2}k_{e3}k_{e4}}}{\Delta_{J_{e1}}\Delta_{J_{e2}}\Delta_{J_{e3}}\Delta_{J_{e4}}}D^{J_{e1}}_{k_{e1}l_{e1}}(g_{e1})...D^{J_{e4}}_{k_{e4}l_{e4}}(g_{e4})w^{J_{e1}}_{l_{e1}}w^{J_{e2}}_{l_{e2}}w^{J_{e3}}_{l_{e3}}w^{J_{e4}}_{l_{e4}}w^{J_{e1}}_{m_{e1}}w^{J_{e2}}_{m_{e2}}w^{J_{e3}}_{m_{e3}}w^{J_{e4}}_{m_{e4}}\prod_v\mathbf{B}_{BC}
\tag{18}
$$

which is not gauge invariant but only gauge covariant.

This would be enough for discarding this variation of the procedure used above as not viable. Nevertheless, this does not lead to any apparent problem when we proceed with the gluing as we did previously. In fact, as it can be easily verified, the additional invariant vectors w do not contribute to the gluing, when the connection is held fixed at the boundary, and the result is again the ordinary Perez-Rovelli version of the Barrett-Crane model. The edge amplitude, i.e. the amplitude for the tetrahedra in the interior of the manifold, is again given by (7). However, the inconsistency appears when we apply the "consistency check" used previously, i.e. when we study the gluing with different boundary conditions. In fact, when the field B is held fixed at the boundary, we have to multiply again the partition function (18) by the additional boundary terms 13, this time imposing the simplicity projections here as well as in (16). The resulting 4-simplex amplitude is:

$$
Z=\prod_f\Delta_{J_f}\prod_{e'}\frac{B^{J_{e'1}J_{e'2}J_{e'3}J_{e'4}}_{k_{e'1}k_{e'2}k_{e'3}k_{e'4}}}{\left(\Delta_{J_{e'1}}\Delta_{J_{e'2}}\Delta_{J_{e'3}}\Delta_{J_{e'4}}\right)^2}\prod_v\mathbf{B}_{BC}
\tag{19}
$$

and the gluing results in an edge amplitude for the interior tetrahedra:

$$A_e = \frac{\Delta_{1234}}{(\Delta_{J_1}\Delta_{J_2}\Delta_{J_3}\Delta_{J_4})^3}. \tag{20}$$

This proves that this model is not consistent, since the result is different for different boundary conditions, and shows also that, as we said above, the "consistency check" is not trivially satisfied by every model.

Considering now the second variation (17), we see that imposing the simplicity constraint this way gives the same result as if we had imposed it directly in the edge amplitude (3), having $A_e(GR) = P_h P_g P_h A_e(BF)$. This, however, breaks the gauge invariance of the edge amplitude, for which we were aiming when we imposed the additional projection P_g. In turn this results into a breaking of the gauge invariance of the 4-simplex amplitude. Because of this we do not explore any further this variation, but rather study directly the simpler case in which we do not impose the projection P_g at all in the edge amplitude. Then we will give more reasons for imposing it.

B. Imposing the projections differently

We then study the model obtained dropping the projection P_g in (3), and not imposing any additional simplicity projection on the D-functions for the exposed edges, since we have just seen that this would lead to inconsistencies (more precisely, projecting the indices referring to the triangles would lead to inconsistencies, while projecting those referring to the tetrahedra would give exactly the same result as not projecting at all, as can be verified).

The edge amplitude replacing (3) is then:

$$A_e = \frac{B^{J_1 J_2 J_3 J_4}_{k_1 k_2 k_3 k_4}}{\sqrt{\Delta_{J_1}\Delta_{J_2}\Delta_{J_3}\Delta_{J_4}}}\, w^{J_1}_{m_1} w^{J_2}_{m_2} w^{J_3}_{m_3} w^{J_4}_{m_4} \tag{21}$$

and the partition function for a single 4-simplex is:

$$Z_{BC} = \sum_{\{J_f\},\{k_{e'}\}} \prod_f \Delta_{J_f} \prod_{e'} \frac{w^{J_1}_{m_1} w^{J_2}_{m_2} w^{J_3}_{m_3} w^{J_4}_{m_4}}{\sqrt{\Delta_{J_{e'1}}\Delta_{J_{e'2}}\Delta_{J_{e'3}}\Delta_{J_{e'4}}}} \prod_v \mathbf{B}_{BC} \left(\prod_{\tilde{e}} D\right), \tag{22}$$

where the D-functions for the exposed edges are contracted not with the Barrett-Crane intertwiners but with the $SO(3)$ invariant vectors w^J_m. Consequently the partition function itself is not an invariant under the group. However, let us go a bit further to see which model results from the gluing. Proceeding to the usual gluing, the resulting edge amplitude for interior tetrahedra is simply:

$$\frac{1}{\Delta_{J_{e'1}}\Delta_{J_{e'2}}\Delta_{J_{e'3}}\Delta_{J_{e'4}}} \tag{23}$$

and the gluing itself looks rather trivial in the sense that in the end it just gives a multiplication of pre-existing factors, with nothing new arising from it. The gluing performed starting from the partition function with the other boundary conditions gives the same result, again only if we do not project the D-functions for the exposed edges.

This is the "factorized" edge amplitude considered in [13], and singled out by the requirement that the passage from $SO(4)$ BF theory to gravity is given by a pure projection operator (how the dual or connection picture changes for the Perez-Rovelli version is shown in [21]). Indeed, we have just seen how this model is obtained using only the implicity projection, and dropping the P_g, which is responsible for making the combined operator $P_g P_h$ not a projector ($P_g P_h P_g P_h \neq P_g P_h$).

On the other hand, the additional projection P_g introduces an additional coupling of the representations for the four triangles forming a tetrahedron. This coupling can be understood algebraically directly from the way the P_g operator acts, since it involves all the four wedges incident to the same edge (see equation (3)), or recalling that the gauge invariance of the edge amplitude (corresponding to the tetrahedra of the simplicial manifold) admits a natural interpretation as the closure constraint for the bivectors B in terms of which we quantize both BF theory and gravity in the Plebanski formulation. This is the constraint that the bivectors assigned to the triangles of the tetrahedron, forced to be simple bivectors because of the simplicity constraint P_h, sum to zero. Thus we can argue more geometrically for the necessity of the P_g projection saying that the model has to describe the geometric nature of the triangles, but also the way they are "coupled" to form "collective structures", like tetrahedra. Not imposing the P_g projection results in a theory of not enough coupled triangles. For this reason we do not consider this as a viable version of the Barrett-Crane model.

But if the P_g projector is necessary, then the procedure of sections II III, giving the Perez-Rovelli version of the Barrett-Crane model, can be seen as the minimal, and most natural, way of constraining BF-theory to get a quantum gravity model. At the same time, exactly because combining the projectors P_h and P_g does not give a projector operator, "non-minimal" models, sharing the same symmetries of the Perez-Rovelli version, and implementing as well the Barrett-Crane constraints, but possibly physically different from it, can be easily constructed, imposing the two projectors more than once. It is easy to verify that, both starting from the partition function for a single 4-simplex with fixed boundary connection or with the B field fixed instead, imposing the combined $P_g P_h$ operator n times ($n \geq 1$), the usual gluing procedure will result in an amplitude for the interior tetrahedra:

$$A_e = \frac{\Delta_{1234}^{2n-1}}{(\Delta_{J_1}\Delta_{J_2}\Delta_{J_3}\Delta_{J_4})^{2n}}. \tag{24}$$

Of course, the same kind of model could be obtained from a field theory over a group, with the usual technology. However, the physical significance of this variation is unclear (apart from the stronger convergence of the partition function, which is quite apparent).

To conclude, let us comment on the De Pietri-Freidel-Krasnov-Rovelli version of the Barrett-Crane model. It seems that there is no natural (or simple) variation of the procedure we used leading to this version of the Barrett-Crane state sum, as we have seen. In other words, starting from the partition function for BF theory, it appears to be no simple way to impose the Barrett-Crane constraints at the quantum level, by means of projector operators as we did, and to obtain a model with an amplitude for the interior tetrahedra of the type:

$$A_e = \frac{1}{\Delta_{1234}} \tag{25}$$

as in [6]. Roughly, the reason can be understood as follows: for each edge, the P_h projection has the effect of giving a factor involving the product of the dimensions of the representations in the denominator, and of course of restricting the allowed representations to the simple ones, while the P_g projector is responsible for having a Barrett-Crane intertwiner for the boundary tetrahedra, which in turn produces the factor Δ_{1234} after the gluing. The imposition of more of these projections in the non-minimal models can only change the power with which these same elements appear in the final partition function, as we said. So it seems that the imposition of these projectors can not create a factor like Δ_{1234} in the denominator, which, if wanted, has apparently to be inserted by hand from the beginning. The un-naturality of this version of the Barrett-Crane model from this point of view can probably be understood noting that in the original field theory over group formulation [6] the imposition of the operator P_h in the kinetic term of the action, giving a kinetic operator that is not a projector anymore, makes the coordinate space (or "connection" [7]) formulation of the partition function highly complicated, and this formulation is the closest analogue of our lattice-gauge-theory-type of derivation. However, an intriguing logical possibility that we think deserves further study is that the edge amplitude (25) may be "expanded in powers of Δ_{1234}", so that it may arise from a (probably asymptotic) series in which the n-th term results from imposing the $P_g P_h$ operator n times with the result shown above. More generally, many different models can be constructed (consistently with different boundary conditions) in this way (combining models with different powers of $P_g P_h$), all based on the simple representations of $SO(4)$ or $SL(2, C)$, having the same fundamental symmetries, and the Perez-Rovelli version of the Barrett-Crane model as the "lowest order" term, with the other orders as "corrections" to it, even if interesting models on their own right. This possibility will be investigated in the future.

V. CONCLUSIONS

We have thus shown how the Barrett-Crane spin foam model for quantum gravity (in the Perez-Rovelli version) can be obtained with a lattice gauge theory type of derivation, with the appropriate boundary terms corresponding to fixing the B (metric) field on the boundary of the manifold. We stress that the correct treatment of the boundary terms and their precise description will be necessary for any concrete application of the spin foam model, like for example the calculation of transition amplitudes between quantum gravity states [22], or the study of black hole physics (e.g computing black hole entropy) [14,23]. We have also described how the gluing between 4-simplices has to be carried out in this context. The result is consistent with the one obtained in [11] fixing the connection field instead. We also explored several variations of this derivation, including one resulting in a class of "non-minimal" models that may turn out to be useful in the future. As a result, the Perez-Rovelli version of the Barrett-Crane model appears to be the simplest consistent outcome of constraining BF theory with a procedure of the kind we used.

ACKNOWLEDGEMENTS

The author is much grateful to E. R. Livine, H. Pfeiffer and R. M. Williams for many discussions and suggestions.

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